# A GENERAL THEORY OF SURFACES

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*Introduction.*—Although the differential geometry of a $k$-dimensional spread or variety $V_k$, embedded in an $n$-dimensional space $S_n$, has received considerable attention, in particular in the case $k = n-1$, the theory of surfaces $V_2$ in the general Euclidean hyperspace $S_n$ has been treated extensively by only three authors, Kommerell, Levi, and Segre,[1] of whom the last was interested in projective properties, the first two in ordinary metric relations. The theory of $V_2$ in $S_n$ does, however, offer points of contact with elementary differential geometry which are fully as illuminating as those developed in the case of $V_{n-1}$ in $S_n$. We have, therefore, undertaken to provide a general theory of surfaces which shall be independent of the number of dimensions of the containing space.

The first thing to be determined in entering on such an extended study is the method of attack. The available methods were four: (1) The ordinary elementary method of starting with the finite equations of the surface and trying to generalize well-known geometric properties. This was followed by Kommerell. (2) The more advanced method of Levi, which depends upon the finite equations only for calculating certain invariants $I$ of rigid motion and further invariants (covariants) $J$ of transformations of parameters upon the surface. (3) The vectorial method of Gibbs and others which bases the work upon the properties of the linear vector function expressing the relation between an infinitesimal displacement in the surface and the infinitesimal change in the unit normal $(n-2)$-space or the unit tangent plane. (4) The method of Ricci's absolute differential calculus, or Maschke's symbolic invariantive method, which develops the theory from the point of view of the fundamental quadratic differential forms defining the surface.

We adopted the last method because of the possibility of sharply differentiating (1) those properties of surfaces which depend on the first fundamental form, and belong to the surface as any one of the infinite class of mutually applicable surfaces, from (2) the properties which follow from the first and second fundamental forms taken together and which belong to the surface as a rigid surface. The adoption of Ricci's rather than Maschke's form of analysis was further dictated by the fact that Ricci[2] had developed the ordinary theory of surfaces consistently by his method, and all his work upon the first part (1), holding without

change in any number of dimensions, could stand without repetition. Our own work therefore need only begin with the study of the second fundamental form. As, however, Ricci's absolute differential calculus is neither well-known nor published in particularly accessible form, it seemed best to prefix to our work an explanation of that calculus.

*The second fundamental form.*—If $\mathbf{y}$ $(u, v)$ be the vector which determines the points on the surface, the second fundamental form is found to be a vector form

$$\Psi = \mathbf{y}_{11}dx_1^2 + 2\,\mathbf{y}_{12}\,dx_1dx_2 + \mathbf{y}_{22}dx_2^2 = \Sigma_{rs}\,\mathbf{y}_{rs}\,dx_rdx_s,$$

where $\mathbf{y}_{11}$, $\mathbf{y}_{12}$, $\mathbf{y}_{22}$ are the second covariant derivatives of $\mathbf{y}$ with respect to $u$ and $v$ and are vectors normal to the surface. If $ds$ denote a differential of arc and

$$\lambda^{(r)} = dx_r/ds, \qquad \lambda'^{(r)} = dx_r/ds', \qquad r = 1, 2,$$

be the differential equations of a system of curves $\lambda$ on the surface and of their orthogonal trajectories, the three vectors $\alpha$, $\beta$, $\mu$,

$$\alpha = \Sigma_{rs}\,\lambda^{(r)}\,\lambda^{(s)}\,\mathbf{y}_{rs}, \qquad \beta = \Sigma_{rs}\,\lambda'^{(r)}\,\lambda'^{(s)}\,\mathbf{y}_{rs},$$
$$\mu = \Sigma_{rs}\,\lambda'^{(r)}\,\lambda^{(s)}\,\mathbf{y}_{rs} = \Sigma_{rs}\,\lambda^{(r)}\,\lambda'^{(s)}\,\mathbf{y}_{rs},$$

are invariant of the parameter system on the surface and are associated with a direction $\lambda$. Further

$$\mathbf{y}_{rs} = \alpha\,\lambda_r\,\lambda_s + \mu\,(\lambda_r\,\lambda' + \lambda'_r\,\lambda_s) + \beta\,\lambda'\,\lambda',$$

where $\lambda_r$ is the dual or reciprocal system to $\lambda^{(r)}$ defined by $\lambda_r = \Sigma_s a_{rs}\,\lambda^{(s)}$ if $ds^2 = \Sigma_{rs}\;a_{rs}\,dx_r\,dx_s$ be the first fundamental form.

The geometric interpretation of $\alpha$ is the normal curvature of the surface in the direction $\lambda$ (i.e., the curvature of the geodesic tangent to $\lambda$); of $\beta$, the same for the perpendicular direction; of $\mu$, the rate of change of a unit vector drawn in the surface perpendicular to the geodesic tangent to $\lambda$. The Gaussian or total curvature $G$ is $G = \alpha\cdot\beta - \mu^2$, the difference of the scalar product of $\alpha$ and $\beta$ and the square of $\mu$; this is independent of the direction $\lambda$ and is one of the prime invariants of the surface. There is a vector $\mathbf{h}$ defined by the equation

$$2\mathbf{h} = \alpha + \beta = \Sigma_{rs}a^{(rs)}\,\mathbf{y}_{rs}, \qquad a^{(rs)} = (-1)^{r+s}a_{rs}/\left|\,a_{rs}\,\right|,$$

which is also independent of the direction $\lambda$ and which we call the mean (vector) curvature of the surface at the point considered; the magnitude of $\mathbf{h}$ is another of the prime invariants of the surface.

If $\mathbf{M}$ be a unit (vector) tangent plane to the surface, and $d\mathbf{M}$ its differential, the second fundamental (vector) form $\Psi$ may be written

$\Psi = -(d\mathbf{y}\cdot\mathbf{M})\cdot d\mathbf{M}$, where the dot denotes the inner product,[3] in complete analogy with the expression $-d\mathbf{y}\cdot d\mathbf{n}$, in terms of the unit normal $\mathbf{n}$, in ordinary surface theory. Moreover, the square of $d\mathbf{M}$ is $(d\mathbf{M})^2 = -Gds^2 + 2\mathbf{h}\cdot\Psi$, and thus gives what may be called the third fundamental form. (In the three-dimensional case, this is associated with the spherical representation.)

*The indicatrix.*—If $\delta = \frac{1}{2}(\alpha - \beta)$ and if accents be used to denote quantities corresponding to a direction at an angle $\theta$ to $\lambda$, we find

$$\mu' = \mu \cos 2\theta - \delta \sin 2\theta, \qquad \delta' = \delta \cos 2\theta + \mu \sin 2\theta.$$

This means that when we turn about a point of the surface the vectors $\mu'$, $\delta'$ describe an ellipse (with center at the extremity of the mean curvature $\mathbf{h}$) of which any two positions of $\mu'$, $\delta'$ are conjugate radii, the eccentric angle between $\mu'$, $\mu$ or $\delta'$, $\delta$ being twice the angle $\theta$. This ellipse we call the indicatrix.[4] The vectors $\alpha'$, $\beta'$ originating at the surface-point $O$ and terminating in the indicatrix describe a cone of normals, Cone I.

The indicatrix is determinative of a large number of properties connected with the curvature of a surface at a point. As the surface is two-dimensional and the indicatrix with the surface-point $O$ determines a normal three-dimensional space, the properties of curvature are as general for a surface $V_2$ in a five-dimensional space $S_5$ as in $S_n$, and the further developments may be given in the assumption that $n = 5$.

Consecutive normal plane three-spaces $N_3$ intersect in a line. These normal lines all pass through a point $O'$ lying upon the perpendicular from $O$, to the plane of the indicatrix and generate a quadric cone, Cone II. The relation between Cones I and II is reciprocal in the sense that each element of either is perpendicular to some tangent plane of the other. Hence if $\Phi$ be the linear vector function which occurs in the equation $\mathbf{r}.\Phi.\mathbf{r} = 0$ of Cone II, referred to its vertex, the equation of Cone I, referred to its vertex, is $\mathbf{r}\cdot\Phi^{-1}\cdot\mathbf{r} = 0$.

The function $\Phi$ may be written as $\Phi = (\mathbf{hh} - \mu\mu - \delta\delta)/a$, where $a = |a_{rs}|$, in terms of the mean curvature $\mathbf{h}$ and a pair of conjugate radii of the indicatrix. $\Phi$ is the self conjugate part of the function

$$\Omega = \begin{vmatrix} \mathbf{y}_{11} & \mathbf{y}_{12} \\ \mathbf{y}_{21} & \mathbf{y}_{22} \end{vmatrix} = \mathbf{y}_{11}\,\mathbf{y}_{22} - \mathbf{y}_{12}\,\mathbf{y}_{21}$$

formed as the determinant (with vector elements) of the second form $\Psi$. We have

$$\Phi = \tfrac{1}{2}\,\mathbf{y}_{11}\,\mathbf{y}_{22} + \tfrac{1}{2}\,\mathbf{y}_{22}\,\mathbf{y}_{11} - \mathbf{y}_{12}\,\mathbf{y}_{21},$$

and the total curvature $G$ is

$$G = \alpha \cdot \beta - \mu^2 = \Phi_S/a = \Omega_S/a,$$

the ratio of the first scalar invariant of $\Phi$, or of the determinant $\Omega$ of the second form $\Psi$, to the determinant $a$ of the first form.

If the surface $V_2$ in $S_5$ be projected successively upon the three spaces $S_3$ determined by the tangent plane and each of three mutually perpendicular normals issuing from $O$ into the normal space $N_3$, the three projections have the (vector) sum of their mean curvatures and the (scalar) sum of their total curvatures equal respectively to the mean curvature and total curvature of the given surface at $O$. There is therefore a cone, Cone III, of normals such that if any element of the cone be chosen as one of the three mutually perpendicular normals, the other two may be chosen in such a way (upon Cone II) that the total curvature of the three projections are respectively $G$, 0, and 0. The equation of this cone is $\mathbf{r} \,.\, (\Phi_S\, I - \Phi) \,.\, \mathbf{r} = 0$, where $I$ is the idemfactor.

*Types of surfaces.*—For minimal surface $\mathbf{h} = 0$, and the indicatrix everywhere reduces to an ellipse in some normal plane and with its center at the surface-point $O$.

For the surface formed by the tangents to a twisted curve, the indicatrix reduces to a segment of a line (described twice) reaching from $O$ to the extremity of the vector 2 $\mathbf{h}$. Such a surface is developable, and all ruled developables are of this type.

For any ruled surface the indicatrix lies in a plane passing through the surface-point $O$, and the ellipse itself passes through $O$. The total curvature of any real ruled surface (other than developable) is negative.

For a surface of revolution, which is formed by revolving a twisted curve parallel to a plane, the indicatrix reduces to a linear segment (described twice) centered at the extremity of $\mathbf{h}$. There is a large variety of developable surfaces (not ruled) of revolution, of which the simplest is perhaps that obtained by revolving the circular helix parallel to a plane containing its axis and a line perpendicular to the three-space in which the helix lies.

For a developable (non-ruled), the indicatrix is tangent to three mutually perpendicular planes through $O$, or, if $n = 4$, to two perpendicular lines through $O$. A developable is not invariant in type under a projective transformation.

The class of surfaces where the indicatrix at each point lies in a plane with $O$ is invariant under projective transformations, and these surfaces have upon them the characteristic lines of Segre.

A special type of surface, which have the indicatrix everywhere a

linear segment noncollinear with **h**, is a simple generalization of ordinary surfaces ($n = 3$) in that by a proper choice of parametric curves the first and second fundamental forms reduce simultaneously to sums of squares. The property is not invariant under a projective transformation.

*Lines upon surfaces.*—Kommerell defines as principal directions those for which the normal curvature is a maximum or minimum, and subsequent authors have followed him. This gives four directions at each point, and they are not simply related to one another. We have chosen to give as a definition of principal directions those for which **h** and $\mu$ are perpendicular. There are two such directions at each point, they are orthogonal and have the properties that the differential tangent planes in the two directions are perpendicular and the values of the rate of change $d\mathbf{M}/ds$ of the tangent plane is numerically a maximum or minimum.

Kommerell defines asymptotic lines, and Levi follows him after restricting the definition to a special case. The definition is not satisfactory, and we define asymptotic lines as those for which **h** and $\alpha$ are perpendicular. The rate of turning of the tangent plane along these directions is the square root of the negative of the total curvature. The differential equations of the lines are $\mathbf{h}\,.\,\Psi = 0$. The lines are bisected by the principal directions.

Our definitions for principal directions and asymptotic lines become illusory for minimum surfaces.

Segre's characteristics exist when the indicatrix lies in a plane with the surface point $O$ and are then those directions which make $\alpha$ tangent to the indicatrix. The asymptotic directions divide the characteristic directions harmonically.

*Development of surfaces.*—In the neighborhood of a point a surface may usually be developed in either of two standard forms.

$$\begin{aligned} & z_1 = \tfrac{1}{2}\,h(x^2 + y^2) + e(x^2 - y^2), && z_2 = \tfrac{1}{2}\,f(x_2 - y^2), \\ & z_3 = \tfrac{1}{2}\,A(x^2 - y^2) + 2\,Bxy, && z_i = 0,\ i > 3, \\ \text{or}\quad & z_2 = \tfrac{1}{2}\,(Ax^2 + 2\,Bxy + Cy^2), && z_2 = \tfrac{1}{2}\,Dx^2,\ z_3 = \tfrac{1}{2}\,Ey^2. \end{aligned}$$

For the first form **h** lies along the axis of $z_1$, the plane of the indicatrix is parallel to the axis of $z_3$, and the axis of $x$ and $y$ are along the principal directions. A hyperplane tangent to the surface at $O$ will cut the surface in real, imaginary, or coincident directions according as it cuts the indicatrix in real, imaginary, or coincident points. The second standard form of the surface has the property that the two perpendicular hyper-

planes $z_2 = 0$ and $z_3 = 0$ cut the surface in orthogonal coincident directions.

Particular interest attaches to the tangent hyperplane perpendicular to the mean curvature **h**. This cuts the surface in the asymptotic directions and the axes of the (degenerate) conic made up of these two directions are the principal directions. The intersection of the surface and this hyperplane has therefore the fundamental properties of the Dupin indicatrix.

For the proof of the geometric results here stated and for the proofs and statements of a large number of others, many of which are entirely new, some only new statements of the results of Levi, Kommerell, or Segre, reference must be made to our complete memoir 'Differential Geometry of Two-dimensional Surfaces in Hyperspace' which will be published in the *Proceedings of the American Academy*, Boston.

[1] Kommerell, Die Krümmung der Zweidimensionalen Gebilde in ebenen Raum von vier Dimensionen, *Dissertation*, Tübingen, 1897, 53 pp; E. E. Levi, Saggio sulla Theoria delle Superficie a due Dimenzioni immersi in un Iperspazio, *Pisa, Ann. R. Scu. Norm*, **10**, 99 pp; C. Segre, Su una Classe di Superficie degl' iperspazî, *Torino, Att. R. Acc. Sci.*, **42**, 1047–1079 (1907).

[2] Ricci, *Lezioni sulla Theoria delle Superficie*, Padova, Drucker, 1898. (Lithographed, edition exhausted.)

[3] The vector analysis used is a modification of the Grassmannian system; see Lewis, *Proc. Amer. Acad. Arts Sci.*, **46**, 165–181 (1910), and Wilson and Lewis, *Ibid.*, **48**, 389–507 (1912).

[4] The corresponding indicatrix for $V_2$ in $S_3$ is not Dupin's but the range of points upon the normal described (twice) by the normal curvature vector $\alpha$.

## DYNAMICAL STABILITY OF AEROPLANES

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The first rational theory of the dynamical stability of aeroplanes is due to Bryan[1] whose work was extended and applied by Bairstow[2] with wind tunnel tests on models.

The utility of such tests in predicting the aerodynamical properties of a full size aeroplane is now well understood and the validity of this application has been repeatedly demonstrated. The late E. T. Busk of the Physical Staff of the Royal Aircraft Factory, England, applied Bairstow's model tests to the design of an aeroplane and recently succeeding in perfecting an inherently stable machine which could be flown 'hands off.' Neither the details of Busk's experiments nor of the type of aeroplane developed by him have been disclosed by the British War Office.